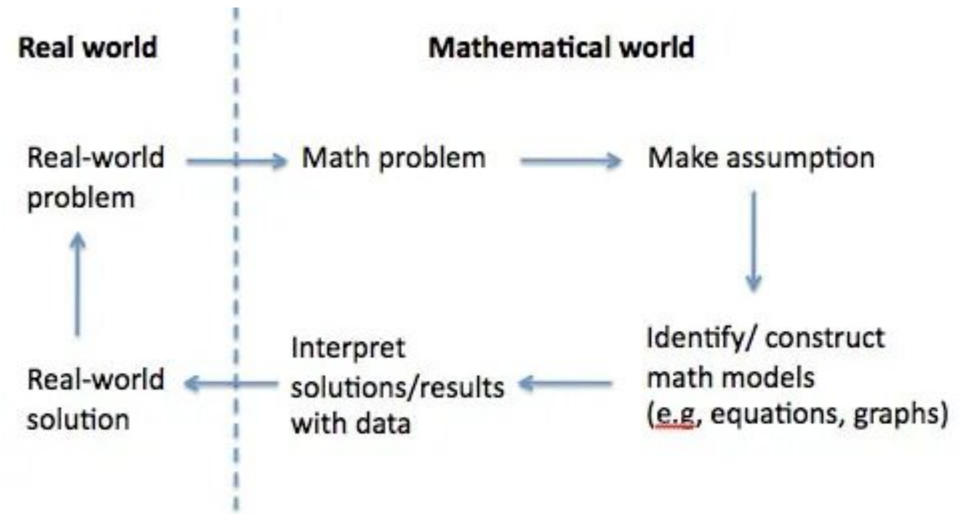
Course Description of Mathematical Modelling

What is mathematical modeling?

While there is no consensus yet as to a precise definition of this term, mathematical modeling is generally understood as the process of applying mathematics to a real world problem with a view of understanding the latter. One can argue that mathematical modeling is the same as applying mathematics where we also start with a real world problem, we apply the necessary mathematics, but after having found the solution we no longer think about the initial problem except perhaps to check if our answer makes sense. This is not the case with mathematical modeling where the use of mathematics is more for understanding the real world problem. The modeling process may or may not result to solving the problem entirely but it will shed light to the situation under investigation. The figure below shows key steps in modeling process.



Mathematical modeling approaches can be categorized into four broad approaches: Empirical models, simulation models, deterministic models, and stochastic models. The first three models can very much be integrated in teaching high school mathematics. The last will need a little stretching.

Empirical modeling involves examining data related to the problem with a view of formulating or constructing a mathematical relationship between the variables in the problem using the available data.

Simulation modeling involve the use of a computer program or some technological tool to generate a scenario based on a set of rules. These rules arise from an interpretation of how a certain process is supposed to evolve or progress.

Deterministic modeling in general involve the use of equation or set of equations to model or predict the outcome of an event or the value of a quantity.

Stochastic modeling takes deterministic modeling one further step. In stochastic models, randomness and probabilities of events happening are taken into account when the equations are formulated. The reason behind this is the fact that events take place with some probability rather than with certainty. This kind of modeling is very popular in business and marketing.

Why is Mathematical Modelling so Important?

Mathematical modelling is valuable in various applications; it gives precision and strategy for problem solution and enables a systematic understanding of the system modelled. It also allows better design, control of a system, and the efficient use of modern computing capabilities.

Knowing the ins and outs of mathematical modelling is a crucial step from theoretical mathematical training to application-oriented mathematical expertise; it also helps the students master the challenges of our modern technological culture.

Looking at the core application of Mathematical Modelling:

We can list some of the modelling applications we understand, at least in some details, with areas involving numerous mathematical experiments. Aside from Engineering and Physics, other various areas have interesting mathematical problems and these include Artificial intelligence, Computer science, Economics, Finance, and the Internet. Mathematical modelling is applicable in Artificial Intelligence (AI), Robotics, speech recognition, optical character recognition, reasoning under computer vision, and image interpretation, among others. It is also important in image processing, realistic computer graphics (ray tracing), and labour data analysis.

Key areas of mathematics useful in Mathematical Modelling:

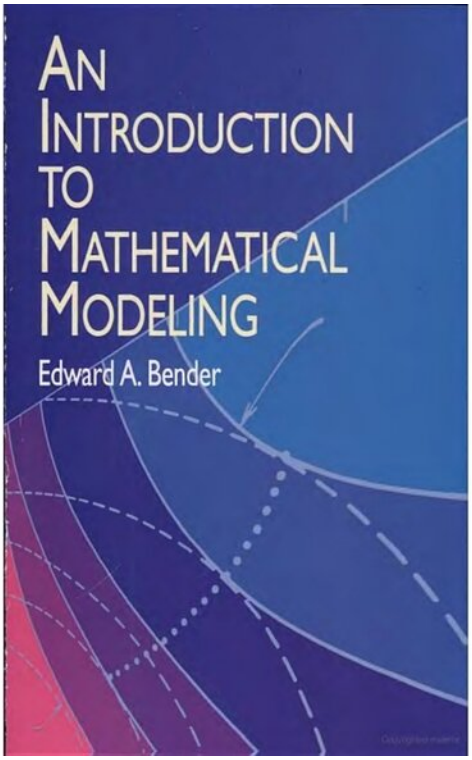
To formulate the basic algorithms for your mathematical formulation, the following are the key mathematical categories: Numerical linear algebra (linear systems of equations, Eigenvalue problems, linear programming, linear optimization, techniques for large, sparse problems), numerical analysis (function evaluation, automatic and numerical differentiation, Interpolation, Approximation Padé, least squares, radial basis functions, special functions, Integration univariate, multivariate, Fourier transform nonlinear systems of equations, optimization and nonlinear programming), numerical data analysis (Visualization 2D and 3D computational geometry), parameter estimation least squares, maximum likelihood, filtering, time correlations, spectral analysis prediction, Classification Time series analysis, signal processing) Categorical Time series, hidden Markov models, random numbers and Monte Carlo methods), and numerical functional analysis (ordinary differential equations, initial value problems, boundary value problems, eigenvalue problems, stability techniques for large problems, partial differential equations finite differences, finite elements, boundary elements, mesh generation, adaptive meshes Stochastic differential equations Integral equations and regularization) and non-numerical algorithms (symbolic methods, computer algebra, sorting, and Compression Cryptography).

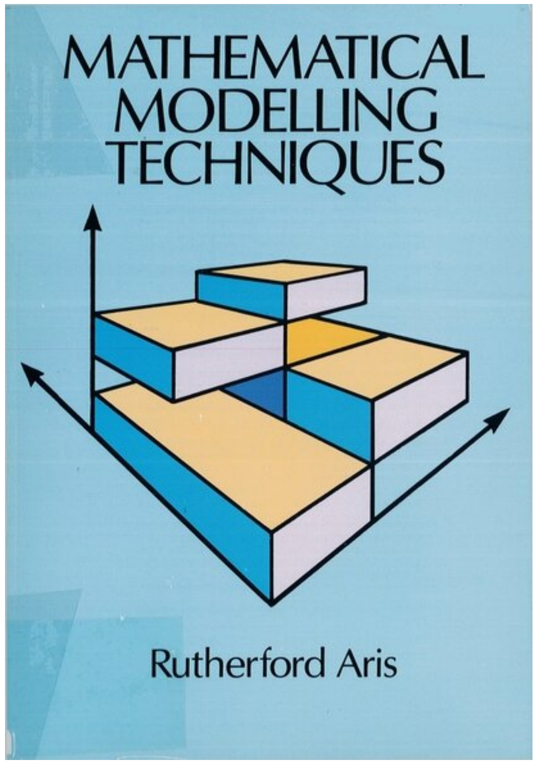
Recommended Textbooks for a Basic Mathematical Modelling Course

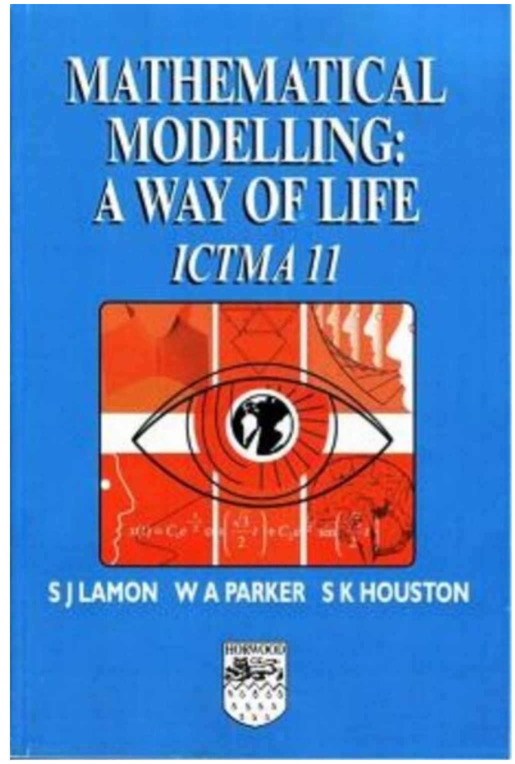
There are numerous mathematical modeling books at different levels and which focus on different topics. We introduce some of these here.

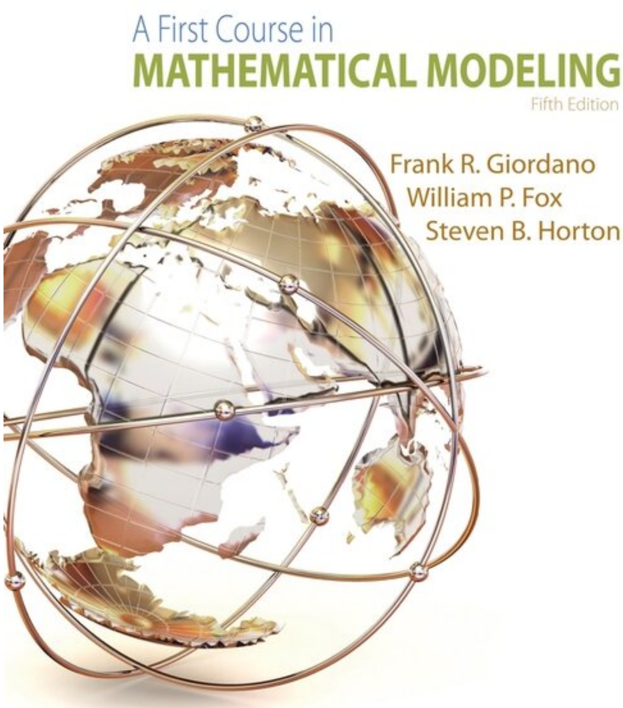
1. **An Introduction to Mathematical Modeling,** *By Edward A. Bender, Published by Dover Publishing Inc, (2012)*
2. **Mathematical Modelling Techniques,** *Revised Edition, By Rutherford Aris, Published by Dover Publishing Inc, (1994)*
3. **Mathematical Modelling, A Way of Life - ICTMA 11,** *(Authors) S J Lamon, W A Parker, S K Houston, 1st Edition, Published By Elsevier (2003)*
4. **A First Course in Mathematical Modeling,** *Author(s): Frank R. Giordano, William P. Fox, Steven B. Horton, Publisher: Cengage Learning, (2013)*
5. **Mathematical Modeling: Applications with GeoGebra**, *Author(s): Jonas Hall, Thomas Lingefjärd Publisher: Published By Wiley, (2016)*

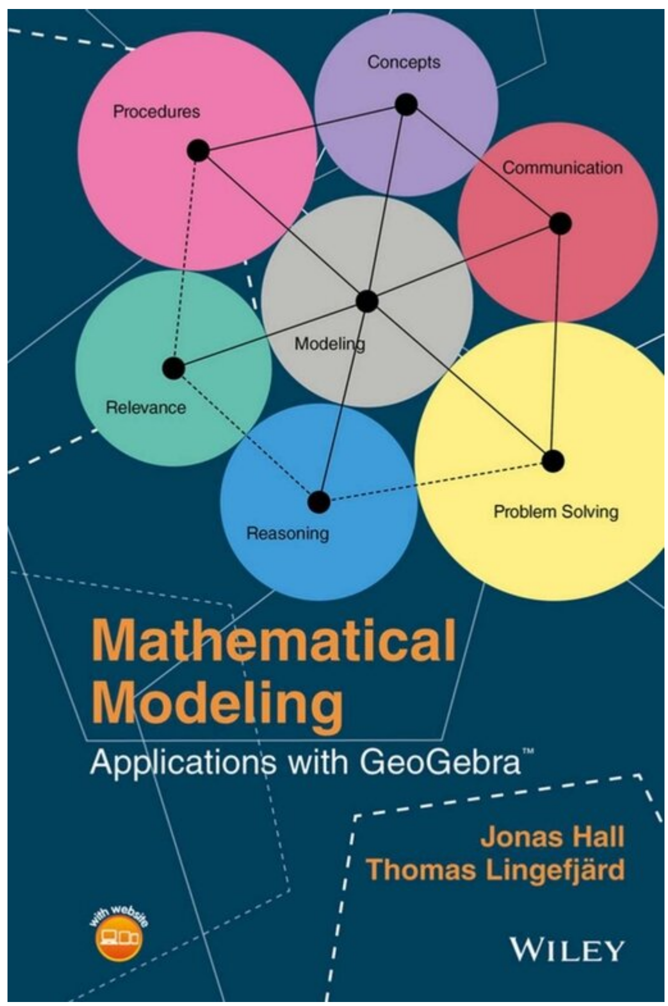
NOTE: For this course our tutors will mostly use (1) as a reference for teaching the contents of this course. However,when needed, they will also choose complementary materials from (2) - (5). Prefaces and Table of Contents of these five textbooks will be given in the next pages, respectively.


AN
INTRODUCTION
TO
MATHEMATICAL
MODELING
Edward A. Bender


MATHEMATICAL
MODELLING
TECHNIQUES
Rutherford Aris


MATHEMATICAL
MODELLING:
A WAY OF LIFE
ICTMA 11
S J LAMON W A PARKER S K HOUSTON


A First Course in
MATHEMATICAL MODELING
Fifth Edition
Frank R. Giordano
William P. Fox
Steven B. Horton


Concepts
Procedures
Communication
Modeling
Relevance
Problem Solving
Reasoning
Mathematical
Modeling
Applications with GeoGebra™
Jonas Hall
Thomas Lingefjärd
WILEY

PREFACE

This book is designed to teach students how to apply mathematics by formulating, analyzing, and criticizing models. It is intended as a first course in applied mathematics for use primarily at an upper division or beginning graduate level. Some course suggestions are given near the end of the preface.

The first part of the book requires only elementary calculus and, in one chapter, basic probability theory. A brief introduction to probability is given in the Appendix. In Part II somewhat more sophisticated mathematics is used.

Although the level of mathematics required is not high, this is not an easy text: Setting up and manipulating models requires thought, effort, and usually discussion—purely mechanical approaches usually end in failure. Since I firmly believe in learning by doing, all the problems require that the student create and study models. Consequently, there are no trivial problems in the text and few very easy ones. Often problems have no single best answer, because different models can illuminate different facets of a problem. Discussion of homework in class by the students is an integral part of the learning process; in fact, my classes have spent about half the time discussing homework. I have also encouraged (or insisted) that homework be done by students working in groups of three or four. We have usually devoted one class period to a single model, both those worked out in the text and those given as problems. I have also required students to report on a model of their own choosing, the amount of originality required depending on the level of the student.

Except for Chapter 6, each section of the text deals with the application of a particular mathematical technique to a range of problems. This lets the students focus more on the modeling. My students and I have enjoyed the variety provided by frequent shifts from one scientific discipline to another. This structure also makes it possible for the teacher to rearrange and delete material as desired; however, Chapter 1 and Section 2.1 should be studied first. Chapter 1 provides a conceptual and philosophical framework. The discussions and problems in Section 2.1 were selected to get students started in mathematical modeling.

Most of the material in this book describes other people's models, frequently arranged or modified to fit the framework of the text, but hopefully without doing violence to the original intentions of the model. I believe all the models deal with questions of real interest: There are no "fake" models created purely to illustrate a mathematical idea, and there are no models that have been so sanitized that they have lost contact with the complexities of the real world. Since I've selected the models, they reflect my interests and knowledge. For this I make no apology—*caveat emptor*.

The models have been chosen to be brief and to keep scientific background at a minimum. While this makes for a more lively and accessible text, it may give the impression that modeling can be done without scientific training and that modeling never leads to involved studies. I thought seriously about counteracting this by adding a few chapters, each one devoted to a specific model. Unable to find a way to do this without sacrificing "learnning by doing," I abandoned the idea.

Course suggestions. On an undergraduate level, the text can be used at a leisurely pace to fill an entire year. It may be necessary to teach some probability theory for Chapter 5, and you may wish to drop Chapter 10. More variety can be obtained by using the text for part of a year and then spending some time on an in-depth study of some additional models—with guest lecturers from the appropriate scientific disciplines if possible. Another alternative is to spend more time on simulation models after Section 5.2 if a computer is available for groups of students to develop their own in-depth models.

Acknowledgments. Particular thanks are due to Norman Herzberg for his many suggestions on the entire manuscript. My students have been invaluable in pointing out discussions and problems that were too muddled or terse to understand. I owe thanks to a variety of people who have commented on parts of the manuscript, suggested models, and explained ideas to me.

I'd appreciate hearing about any errors, difficulties encountered, suggestions for additional material, or anything else that might improve future editions of this book.

EDWARD A. BENDER

La Jolla, California
August 1977

CONTENTS

Preface

> "Par ma foi! il y a plus de quarante ans que
> je dis de la prose sans que j'en susse rien,
> et je vous suis le plus obligé du monde de
> m'avoir appris cela."
>
> M. Jourdain in Moliere's "Le Bourgeois Gentilhomme" (Act. II Sc. IV)

The original title under which these notes were written--"Notes toward the definition of the craft of mathematical modelling"--was somewhat long-winded and perhaps, by reason of its allusion, a shade pretentious. It had however the merit of greater precision and conveyed the tentative spirit in which these notes are put forward for the criticism of a larger public. The whole activity of mathematical modelling has blossomed forth into such a multitude of areas in the last few years (witness a 1st International Conference with 2646 pages of Proceedings [199]) that there is indeed a need to define it in the sense of seeking out its boundary and exploring its interior as well as of discovering its structure and essential nature. The time is not yet ripe for a magisterial survey, which would in any case demand an abler pen than mine, but I believe it can be approached from the angle of craftsmanship. It is a commonplace in educational circles that it is comparatively easy to teach the method of solution of a standard mathematical equations, but much harder to communicate the ability to formulate the equations adequately and economically. With the notable exception of Lin and Segel's [223], and Haberman's book [213] and the papers of Hammersley [80,81,82] few publications pay much attention to the little things that the experienced mathematical modeller does, almost by instinct. It would therefore seem to be worthwhile to try and set down some of these notions in the interests of the craft and with the hope that it will stimulate further discussion and development. There is manifestly a danger here, for it may be only the MM. Jourdains who will be vastly excited to learn that they have been talking prose all their lives. Nevertheless I hope that some of my peers and betters will find the subject worthy of their attention.

Later iterations of this effort will demand a wealth of examples drawn from all branches of the physical and social sciences. In this first attempt I have chosen three physical examples to serve the illustration of many points. These examples--the packed bed, the chromatographic column and the stirred tank--are given in detail in the appendices. They are in some sense fold-out maps to the text though they cannot be presented as such. (Each has its own nomenclature which is listed at the end of its discussion; the nomenclature for other examples is introduced in situ.) These examples and those introduced at various points of the text are often connected with the mathematical theory of chemical reactors. I make no apology for this; the field is a rich one that has stimulated some of the work of the best applied mathematicians who have used a reactor like a stalking-horse under cover of which to shoot their wit. Its problems are challenging, yet from the modelling point of view they do not demand any great knowledge of chemistry or of engineering and so are accessible to all.

Many of the notions I have advanced here and the order I have tried to impose on the subject are quite tentative and I shall appreciate any comments and criticism. I have already benefitted from interaction with colleagues, both faculty and students, at Caltech and it is one of the virtues of

Pitman's Research Notes for Mathematics series that it quickly submits ideas to a wider public.

To the California Institute of Technology I am vastly indebted for a term as a Sherman Fairchild Distinguished Scholar in the fall of 1976, under conditions of such generous hospitality that the fruits of such a tenure can never be worthy of the opportunity. At the risk of overlooking someone, I would like to thank in particular (and in alphabetical order) Cohen, Gavalas, Keller, Pings, Seinfeld and Weinberg. Yolande Johnson did a splendid job of the first draft of these notes that was prepared at Caltech, being helped by Sharon ViGario in the last minute rush. The final version was typed by Shirley Tabis who met the exacting requirements of camera readiness with great skill and dispatch. I am most grateful to all of them.

Contents

Table of Contents

Section D: Research

Section E: Perspectives

PREFACE

Mathematical Modelling: A Way of Life

Those who teach mathematical modelling at the university level and those who use it to solve problems in a wide variety of disciplines, speak of mathematical modelling as a "way of life." This phrase refers to their worldview, their habits of mind, and their dependence on the power of mathematics to describe, explain, predict, and control real phenomena. The expression suggests that mathematics is indispensable as a way of knowing about the world in which they live and about the complex phenomena that affect the quality of their lives. Everything turns into mathematics.

The great difficulty that students face when they study mathematical modelling at the university suggests that it is nearly impossible to adopt this new way of looking at the world so late in one's education. Without any prior experience in building, interpreting and applying mathematical models, it is difficult to imagine that some students will ever see modelling as a "way of life." It is clearly not enough that students go through the motions of educating themselves by accumulating and remembering a storehouse of unconnected bits and facts. If students do not develop the spirit of scientific investigation---longing to know and to understand, questioning all sorts phenomena, conducting logical and systematic investigations, considering premises, and predicting and explaining consequences—they must be helplessly obedient to emotions, pressures, influences, and the authority of other people. At best, they will be reactive and defensive in the face of every problem or crisis that occurs.

Accordingly, one of the chief goals of ICTMA 11 is to explore the ways in which teachers at all levels of schooling may provide opportunities for their students to model a variety of real phenomena in ways that are appropriately matched to the students' mathematical backgrounds and interests. Conference participants were invited to examine from a variety of perspectives what it means to move beyond the efficient transmission of content in the mathematics classroom, toward creating a classroom atmosphere that conveys critical values, shapes useful processes, and rewards powerful thinking. This volume contains 23 contributions to ICTMA 11, many of which address the problems of helping school students to adopt mathematical modeling as a way of life.

ICTMA 11 has the distinction of marking at least two "firsts." As we write this preface, it is three months before conference convenes in Milwaukee, Wisconsin, USA. The presenters/authors and the editors have worked intensively during the year preceding the conference to prepare manuscripts so that conference participants can receive this book when they arrive in Milwaukee. In part, this effort is a response to the ever-lengthening time period between the end of a conference and the book's publication—in some cases, almost two years. We suspect that looking ahead after the conference may be more motivating and productive than looking back. We hope that the extensive review, feedback, and revision process that has already taken place will make for interesting and well-prepared presentations at the conference, and that discussion of these papers in Milwaukee will stimulate ideas and fuel follow-up studies well in advance of the ICTMA 12 meeting in London.

Unfortunately, ICTMA 11 is also the first of our conferences for which participants have had to make travel plans during wartime. Because of the war in Iraq, the SARS epidemic, and the resulting difficulties with the airline companies, conference registrations are considerably lower than ever before and we are very grateful to our publisher, Ellis Horwood of Horwood Publishing, for producing a paperback book. We appreciate not only his willingness to provide this war-time edition, but the consistent support he has shown ICTMA for the last twenty years.

We are grateful to all of the reviewers who freely gave of their time and talents to help the authors and the editors prepare manuscripts for publication. We express our gratitude to Mrs. Pamela Entrikin for her hard work in organizing ICTMA 11, to Marquette University for hosting the conference, and to our friends and corporate sponsors.

Sue Lamon, Bill Parker, and Ken Houston

Contents

7 Optimization of Discrete Models 242

8 Modeling Using Graph Theory 298

10 Game Theory 378

11 Modeling with a Differential Equation 458

13 Optimization of Continuous Models 569

A APPENDIX

Chapters on Website:

14 Dimensional Analysis and Similitude 1

Additional Contents on Website

1 UMAP Modules (COMAP)

The Undergraduate Applications in Mathematics modules (UMAPs) are developed and produced by the Consortium for Mathematics and Its Applications, Inc. (800-772-6627, **www.comap.com**). UMAPs are particularly suited to supplement the modeling course we propose. The following UMAPs are referenced as projects, further reading, or sources for additional problems and are provided on the CD for easy access.

UMAP 60–62 *The Distribution of Resources*
UMAP 67 *Modeling the Nervous System*
UMAP 69 *The Digestive Process of Sheep*
UMAP 70 *Selection in Genetics*
UMAP 73 *Epidemics*
UMAP 74 *Tracer Methods in Permeability*
UMAP 75 *Feldman's Model*
UMAP 208 *General Equilibrium: I*
UMAP 211 *The Human Cough*
UMAP 232 *Kinetics of Single-Reactant Reactions*
UMAP 234 *Radioactive Chains: Parents and Daughters*
UMAP 269 *Monte Carlo: The Use of Random Digits*
UMAP 270 *Lagrange Multipliers: Applications to Economics*
UMAPs 292–293 *Listening to the Earth: Controlled Source Seismology*
UMAP 294 *Price Discrimination and Consumer Surplus*
UMAP 303 *The Diffusion of Innovation in Family Planning*
UMAP 304 *Growth of Partisan Support I*
UMAP 305 *Growth of Partisan Support II*
UMAP 308 *The Richardson Arms Race Model*
UMAP 311 *The Geometry of the Arms Race*
UMAP 321 *Curve Fitting via the Criterion of Least Squares*
UMAP 322 *Difference Equations with Applications*
UMAP 327 *Adjusted Rates: The Direct Rate*
UMAP 331 *Ascent-Descent*
UMAP 332 *The Budgetary Process I*
UMAP 333 *The Budgetary Process II*
UMAP 340 *The Poisson Random Process*
UMAP 341 *Five Applications of Max-Min Theory from Calculus*
UMAP 376 *Differentiation, Curve Sketching, and Cost Functions*
UMAP 453 *Linear Programming in Two Dimensions: I*
UMAP 454 *Linear Programming in Two Dimensions: II*
UMAP 468 *Calculus of Variations with Applications in Mechanics*
UMAP 506 *The Relationship Between Directional Heading of an Automobile and Steering Wheel Deflection*
UMAP 517 *Lagrange Multipliers and the Design of Multistage Rockets*
UMAP 518 *Oligopolistic Competition*
UMAP 520 *Random Walks: An Introduction to Stochastic Processes*
UMAP 522 *Unconstrained Optimization*
UMAP 526 *Dimensional Analysis*
UMAP 539 *I Will If You Will… Individual Threshold and Group Behavior*

UMAP 551 *The Pace of Life: An Introduction to Empirical Model Fitting*
UMAP 564 *Keeping Dimensions Straight*
UMAP 590 *Random Numbers*
UMAP 610 *Whales and Krill: A Mathematical Model*
UMAP 628 *Competitive Hunter Models*
UMAP 675 *The Lotka-Volterra Predator-Prey Model*
UMAP 684 *Linear Programming via Elementary Matrices*
UMAP 709 *A Blood Cell Population Model, Dynamical Diseases, and Chaos*
UMAP 737 *Geometric Programming*
UMAP 738 *The Hardy-Weinberg Equilibrium*

2 Past Modeling Contest Problems

Past contest problems are excellent sources for modeling projects or sources to design a problem. On the CD we provide links to electronic copies of all contest problems:

Mathematical Contest in Modeling (MCM): *1985–2012*
Interdisciplinary Contest in Modeling (ICM): *1997–2012*
High School Contest in Modeling (HiMCM): *1998–2012*

3 Interdisciplinary Lively Applications Projects (ILAPs)

Interdisciplinary Lively Applications Projects (ILAPs) are developed and produced by the Consortium for Mathematics and Its Applications, Inc., COMAP (800-772-6627, **www.comap.com**). ILAPs are codesigned with a partner discipline to provide in-depth model development and analysis from both a mathematical perspective and that of the partner discipline. We find the following ILAPs to be particularly well suited for the course we propose:

- Car Financing
- Choloform Alert
- Drinking Water
- Electric Power
- Forest Fires
- Game Theory
- Getting the Salt Out
- Health Care
- Health Insurance Premiums
- Hopping Hoop
- Bridge Analysis
- Lagniappe Fund
- Lake Pollution
- Launch the Shuttle
- Pollution Police
- Ramps and Freeways
- Red & Blue CDs
- Drug Poisoning
- Shuttle
- Stocking a Fish Pond
- Survival of Early Americans
- Traffic Lights
- Travel Forecasting
- Tuition Prepayment
- Vehicle Emissions
- Water Purification

4 Technology and Software

Mathematical modeling often requires technology in order to use the techniques discussed in the text, the modules, and ILAPs. We provide extensive examples of technology using spreadsheets (Excel), computer algebra systems (Maple®, Mathematica®, and Matlab®), and the graphing calculator (TI). Application areas include:

- Difference Equations
- Model Fitting
- Empirical Model Construction
- Divided Difference Tables
- Cubic Splines
- Monte Carlo Simulation Models
- Discrete Probabilistic Models
- Reliability Models
- Linear Programming
- Golden Section Search
- Euler's Method for Ordinary Differential Equations
- Euler's Method for Systems of Ordinary Differential Equations
- Nonlinear Optimization

5 Technology Labs

Examples and exercises designed for student use in a laboratory environment are included, addressing the following topics:

- Difference Equations
- Proportionality
- Model Fitting
- Empirical Model Construction
- Monte Carlo Simulation
- Linear Programming
- Discrete Optimization Search Methods
- Ordinary Differential Equations
- Systems of Ordinary Differential Equations
- Continuous Optimization Search Methods

Preface

To facilitate an early initiation of the modeling experience, the first edition of this text was designed to be taught concurrently or immediately after an introductory business or engineering calculus course. In the second edition, we added chapters treating discrete dynamical systems, linear programming and numerical search methods, and an introduction to probabilistic modeling. Additionally, we expanded our introduction of simulation. In the third edition we included solution methods for some simple dynamical systems to reveal their long-term behavior. We also added basic numerical solution methods to the chapters covering modeling with differential equations. In the fourth edition, we added a new chapter to address modeling using graph theory, an area of burgeoning interest for modeling contemporary scenarios. Our chapter introduces graph theory from a modeling perspective to encourage students to pursue the subject in greater detail. We also added two new sections to the chapter on modeling with a differential equation: discussions of separation of variables and linear equations. Many of our readers had expressed a desire that analytic solutions to first-order differential equations be included as part of their modeling course. In the fifth edition, we have added two new chapters, Chapter 9, Modeling with Decision Theory and Chapter 10, Game Theory. Decision theory, also called decision analysis, is a collection of mathematical models to assist people in choosing among alternative courses of action in complex situations involving chance and risk. Game Theory then expands decision theory to include decisions where the payoff for the decision maker depends upon one or more additional decision makers. We present both total and partial conflict games.

The text is organized into two parts: Part One, Discrete Modeling (Chapters 1–10 and Chapter 14), and Part Two, Continuous Modeling (Chapters 11–13 and Chapter 15). This organizational structure allows for teaching an entire modeling course that is based on Part One and does not require the calculus. Part Two then addresses continuous models including models requiring optimization and models using differential equations that can be presented concurrently with freshman calculus. The text gives students an opportunity to cover all phases of the mathematical modeling process. The CD-ROM accompanying the text contains software, additional modeling scenarios and projects, and a link to past problems from the Mathematical Contest in Modeling. We thank Sol Garfunkel and the COMAP staff for their support of modeling activities that we refer to under Resource Materials below.

Goals and Orientation

The course continues to be a bridge between the study of mathematics and the applications of mathematics to various fields. The course affords the student an early opportunity to see how the pieces of an applied problem fit together. The student investigates meaningful and practical problems chosen from common experiences encompassing many academic disciplines, including the mathematical sciences, operations research, engineering, and the management and life sciences.

This text provides an introduction to the entire modeling process. Students will have opportunities to practice the following facets of modeling and enhance their problem-solving capabilities:

1. *Creative and Empirical Model Construction:* Given a real-world scenario, the student learns to identify a problem, make assumptions and collect data, propose a model, test the assumptions, refine the model as necessary, fit the model to data if appropriate, and analyze the underlying mathematical structure of the model to appraise the sensitivity of the conclusions when the assumptions are not precisely met.
2. *Model Analysis:* Given a model, the student learns to work backward to uncover the implicit underlying assumptions, assess critically how well those assumptions fit the scenario at hand, and estimate the sensitivity of the conclusions when the assumptions are not precisely met.
3. *Model Research:* The student investigates a specific area to gain a deeper understanding of some behavior and learns to use what has already been created or discovered.

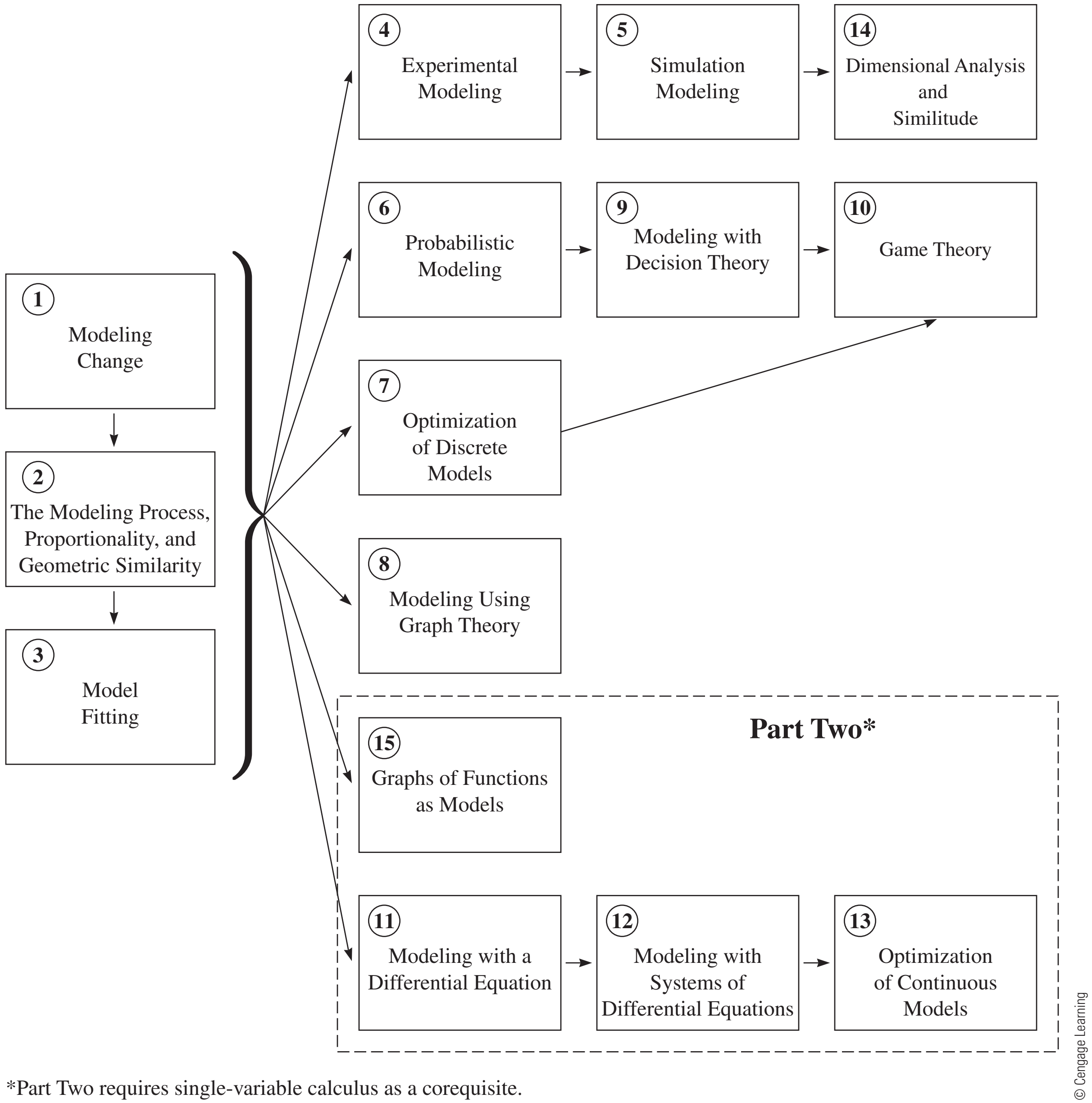
Student Background and Course Content

Because our desire is to initiate the modeling experience as early as possible in the student's program, the only prerequisite for Chapters 11, 12 and 13 is a basic understanding of single-variable differential and integral calculus. Although some unfamiliar mathematical ideas are taught as part of the modeling process, the emphasis is on using mathematics that the students already know after completing high school. This is especially true in Part One. The modeling course will then motivate students to study the more advanced courses such as linear algebra, differential equations, optimization and linear programming, numerical analysis, probability, and statistics. The power and utility of these subjects are intimated throughout the text.

Further, the scenarios and problems in the text are not designed for the application of a particular mathematical technique. Instead, they demand thoughtful ingenuity in using fundamental concepts to find reasonable solutions to ''open-ended'' problems. Certain mathematical techniques (such as Monte Carlo simulation, curve fitting, and dimensional analysis) are presented because often they are not formally covered at the undergraduate level. Instructors should find great flexibility in adapting the text to meet the particular needs of students through the problem assignments and student projects. We have used this material to teach courses to both undergraduate and graduate students—and even as a basis for faculty seminars.

Organization of the Text

The organization of the text is best understood with the aid of Figure 1. The first ten chapters (and Chapter 14) constitute Part One and, require only precalculus mathematics as a



*Part Two requires single-variable calculus as a corequisite.

Figure 1

Chapter organization and progression

prerequisite. We begin with the idea of modeling *change* using simple finite difference equations. This approach is quite intuitive to the student and provides us with several concrete models to support our discussion of the modeling process in Chapter 2. There we classify models, analyze the modeling process, and construct several proportionality models or submodels that are then revisited in the next two chapters. In Chapter 3 the student is presented with three criteria for fitting a specific type of curve to a collected data set, with emphasis on the least-squares criterion. Chapter 4 addresses the problem of capturing the trend of a collected set of data. In this empirical construction process, we begin with fitting simple one-term models approximating collected data sets and then progress to more sophisticated interpolating models, including polynomial smoothing models and cubic splines. Simulation models are discussed in Chapter 5. An empirical model is fit to some collected data, and then Monte Carlo simulation is used to duplicate the behavior being investigated. The presentation motivates the eventual study of probability and statistics.

Chapter 6 provides an introduction to probabilistic modeling. The topics of Markov processes, reliability, and linear regression are introduced, building on scenarios and analysis presented previously. Chapter 7 addresses the issue of finding the best-fitting model using the other two criteria presented in Chapter 3. Linear programming is the method used for finding the best model for one of the criteria, and numerical search techniques can be used for the other. The chapter concludes with an introduction to numerical search methods, including the Dichotomous and Golden Section methods. Chapters 9 and 10 treat decision making under risk and uncertainty, with either one decision maker (Chapter 9) or two or more decision makers (Chapter 10). Part One then skips to Chapter 14, which is devoted to dimensional analysis, a topic of great importance in the physical sciences and engineering.

Part Two is dedicated to the study of continuous models. In Chapters 11 and 12 we model dynamic (time varying) scenarios. These chapters build on the discrete analysis presented in Chapter 1 by now considering situations where time is varying continuously. Chapter 13 is devoted to the study of continuous optimization. Chapter 15 treats the construction of continuous graphical models and explores the sensitivity of the models constructed to the assumptions underlying them. Students get the opportunity to solve continuous optimization problems requiring only the application of elementary calculus and are introduced to constrained optimization problems as well.

Student Projects

Student projects are an essential part of any modeling course. This text includes projects in creative and empirical model construction, model analysis, and model research. Thus we recommend a course consisting of a mixture of projects in all three facets of modeling. These projects are most instructive if they address scenarios that have no unique solution. Some projects should include *real* data that students are either given or can *readily* collect. A combination of individual and group projects can also be valuable. Individual projects are appropriate in those parts of the course in which the instructor wishes to emphasize the development of individual modeling skills. However, the inclusion of a group project early in the course gives students the exhilaration of a ‘‘brainstorming’’ session. A variety of projects is suggested in the text, such as constructing models for various scenarios, completing UMAP[1] modules, or researching a model presented as an example in the text or class. It is valuable for each student to receive a mixture of projects requiring either model construction, model analysis, or model research for variety and confidence building throughout the course. Students might also choose to develop a model in a scenario of particular interest, or analyze a model presented in another course. We recommend five to eight short projects in a typical modeling course. Detailed suggestions on how student projects can be assigned and used are included in the Instructor's Manual that accompany this text.

In terms of the number of scenarios covered throughout the course, as well as the number of homework problems and projects assigned, we have found it better to pursue a few that are developed carefully and completely. We have provided many more problems and projects than can reasonably be assigned to allow for a wide selection covering many different application areas.

Resource Materials

We have found material provided by the Consortium for Mathematics and Its Application (COMAP) to be outstanding and particularly well suited to the course we propose. Individual modules for the undergraduate classroom, UMAP Modules, may be used in a variety of ways. First, they may be used as instructional material to support several lessons. In this mode a student completes the self-study module by working through its exercises (the detailed solutions provided with the module can be conveniently removed before it is issued). Another option is to put together a block of instruction using one or more UMAP modules suggested in the projects sections of the text. The modules also provide excellent sources for "model research," because they cover a wide variety of applications of mathematics in many fields. In this mode, a student is given an appropriate module to research and is asked to complete and report on the module. Finally, the modules are excellent resources for scenarios for which students can practice model construction. In this mode the instructor writes a scenario for a student project based on an application addressed in a particular module and uses the module as background material, perhaps having the student complete the module at a later date. The CD accompanying the text contains most of the UMAPs referenced throughout. Information on the availability of newly developed interdisciplinary projects can be obtained by writing COMAP at the address given previously, calling COMAP at 1-800-772-6627, or electronically: order@comap.com.

A great source of student-group projects are the Mathematical Contest in Modeling (MCM) and the Interdisciplinary Contest in Modeling (ICM). These projects can be taken from the link provided on the CD and tailored by the instructor to meet specific goals for their class. These are also good resources to prepare teams to compete in the MCM and ICM contests. The contest is sponsored by COMAP with funding support from the National Security Agency, the Society of Industrial and Applied Mathematics, the Institute

[1] UMAP modules are developed and distributed through COMAP, Inc., 57 Bedford Street, Suite 210, Lexington, MA 02173.

for Operations Research and the Management Sciences, and the Mathematical Association of America. Additional information concerning the contest can be obtained by contacting COMAP, or visiting their website at **www.comap.com**.

The Role of Technology

Technology is an integral part of doing mathematical modeling with this textbook. Technology can be used to support the modeling of solutions in all of the chapters. Rather than incorporating lots of varied technologies into the explanations of the models directly in the text, we decided to include the use of various technology on the enclosed CD. There the student will find templates in Microsoft® Excel®, Maple®, Mathematica®, and Texas Instruments graphing calculators, including the TI-83 and 84 series.

We have chosen to illustrate the use of *Maple* in our discussion of the following topics that are well supported by Maple commands and programming procedures: difference equations, proportionality, model fitting (least squares), empirical models, simulation, linear programming, dimensional analysis, modeling with differential equations, modeling with systems of differential equations, and optimization of continuous models. Maple worksheets for the illustrative examples appearing in the referenced chapters are provided on the CD.

Mathematica was chosen to illustrate difference equations, proportionality, model fitting (least squares), empirical models, simulation, linear programming, graph theory, dimensional analysis, modeling with differential equations, modeling with systems of differential equations, and optimization of continuous models. Mathematica worksheets for illustrative examples in the referenced chapters are provided on the CD.

Excel is a spreadsheet that can be used to obtain numerical solutions and conveniently obtain graphs. Consequently, Excel was chosen to illustrate the iteration process and graphical solutions to difference equations. It was also selected to calculate and graph functions in proportionality, model fitting, empirical modeling (additionally used for divided difference tables and the construction and graphing of cubic splines), Monte Carlo simulation, linear programming (Excel's Solver is illustrated), modeling with differential equations (numerical approximations with both the Euler and the Runge-Kutta methods), modeling with systems of differential equations (numerical solutions), and Optimization of Discrete and Continuous Models (search techniques in single-variable optimization such as the dichotomous and Golden Section searches). Excel worksheets can be found on the website.

The *TI calculator* is a powerful tool for technology as well. Much of this textbook can be covered using the TI calculator. We illustrate the use of TI calculators with difference equations, proportionality, modeling fitting, empirical models (Ladder of Powers and other transformations), simulation, and differential equations (Euler's method to construct numerical solutions).

Acknowledgments

It is always a pleasure to acknowledge individuals who have played a role in the development of a book. We are particularly grateful to Brigadier General (retired) Jack M. Pollin and Dr. Carroll Wilde for stimulating our interest in teaching modeling and for support and guidance in our careers. We acknowledge and are especially grateful for the support and encouragement of Sol Garfunkel and the entire COMAP staff who have been especially helpful in all five editions: they are true pioneers and champions of mathematical modeling at all levels. We're indebted to many colleagues for reading the first edition manuscript and suggesting modifications and problems: Rickey Kolb, John Kenelly, Robert Schmidt, Stan Leja, Bard Mansager, and especially Steve Maddox and Jim McNulty. We especially thank Maurice D. Weir for his contribution as an author to previous editions. We also especially thank Richard West for his role as a reviewer for the fifth edition.

We are indebted to a number of individuals who authored or co-authored UMAP materials that support the text: David Cameron, Brindell Horelick, Michael Jaye, Sinan Koont, Stan Leja, Michael Wells, and Carroll Wilde. In addition, we thank Solomon Garfunkel and the entire COMAP staff for their cooperation on all five editions of this text. We also thank Tom O'Neil and his students for their contributions to the CD and for helpful suggestions in support of modeling activities. We would like to thank Dr. Amy H. Erickson for her many contributions to the CD and website.

Thanks to the following reviewers of the fifth edition: John Dossey, Robert Burks, and Richard West.

The production of any mathematics text is a complex process, and we have been especially fortunate in having a superb and creative production staff at Brooks/Cole and Cengage for the production of each edition of the text. We would like to thank everyone from Cengage who worked with us on this edition, especially Molly Taylor, our Acquisitions Editor, and Shaylin Walsh-Hogan, our Assistant Editor. Thanks also to Prashanth Kamavarapu and PreMediaGlobal for production service.

Frank R. Giordano
William P. Fox
Steven B. Horton

CONTENTS

PREFACE

Welcome to the world of GeoGebra! GeoGebra is a mathematical environment that will allow you to work with graphing, dynamic 2D and 3D geometry, dynamic and symbolic algebra, spreadsheets, probability, complex numbers, differential equations, dynamic text, fitting functions of any kind to data, and so forth. All representations of mathematical objects are linked and allow you to view, experiment with, and analyze problems and situations in a laboratory-like setting. You can build a geometrical model in one window, animate it, and collect data to a spreadsheet or directly to a graph. GeoGebra works on computers, tablets, and iPhones, under multiple operating systems and is free to use for noncommercial purposes. It can be used in over 65 different languages delivered from a menu option.

Welcome also to the world of mathematical modeling in high school (grades 10–12). With computer tools such as GeoGebra and Wolfram Alpha you can now expand the relatively modest modeling normally done in class to include more real-life situations and solve more interesting and difficult problems. While the computer does the necessary calculations and graphing, with GeoGebra your students will learn the process of translating problem situations to the mathematical language that the computer needs to be able to work efficiently. They will also learn to interpret the results that the computer gives them and draw sensible conclusions from them.

These competencies—translating problem situations to mathematical language suitable for computer processing and analyzing, reasoning about and presenting results—are more important in a modern world than performing specific algorithmic calculations. After all, no one today complains about the fact that we no longer teach the manual algorithm for calculating square roots.

This book came about because the authors strongly believe that modeling lies at the heart of mathematics. And in order to model interesting problems, you need powerful tools. The strong user community and fast development of GeoGebra made it the tool of choice. With it, we were able to explore a multitude of modeling situations, some quite simple and some involving systems of differential equations, something not normally taught in high school but rendered possible with the use of computers.

In Sweden, modeling is one of seven competencies the students are to develop in mathematics education. On the cover of this book, we have symbolically placed the modeling competence at the center of the other competencies, indicating our belief that modeling is absolutely central in mathematics.

In this English translation of the book, we have restructured our material so that it now comes in order of mathematical content. Thus we explore linear models, non-linear models, models requiring calculus and differential equations, and discrete and geometrical models. It is then up to you to decide what problems in this book might be suitable for what courses and what students you teach in your school.

This book should be useful for both experienced teachers and for those students who are in teacher education programs. In addition, students of mathematical modeling at starting level university courses may find this book useful, as may anyone wishing to learn GeoGebra really well. Indeed, the book is intended to be a handy reference on both modeling and GeoGebra, for the reader to keep and return to look up the details of problem solving, modeling, and GeoGebra techniques. If you are new to GeoGebra, we suggest that you read Appendix A, *An Introduction to GeoGebra*, first before continuing with the rest of the book.

Several digital resources are available with this book. On the book's website you will find a collection of all the GeoGebra files used in the book's problems, a list of clickable links, and some screencasts showing basic techniques.

Last we wish to acknowledge the continued support we have received from the GeoGebra community, which has encouraged us to write this book, and we wish to thank our families for being, above all else, patient.

Sweden, May 2016

JONAS HALL AND
THOMAS LINGEFJÄRD