## **Description of Measure & Integration**

The concepts from the theory of measure and integration are vital to any advanced courses in analysis and its applications, specially in the applications of Functional Analysisto other are as such as Harmonic Analysis, Partial Differential Equations and Integral Equations, and in the theoretical investigations in applied mathematics. Therefore, an early introduction to such concepts become essential in the masters program in mathematics.

This course is an attempt towards that goal using minimum background in mathematical analysis. It is essentially an updated version of the notes which we have been using for teaching courses on measure and integration theory several times for 30 years, during 1987–2017. The topics covered in this course are standard ones. However, the students will definitely find that the presentation of the concepts and results is different from the standard courses.

It starts by a short introduction on Riemann integration to motivate the necessity of the concept of integration of functions that are more general than those allowed in Riemann integration and then introduces the concept of Lebesgue measurable sets that is more general than the concept of intervals. Once we have this family of measurable sets, and the concept of a Lebesgue measure, it becomes almost obvious that one need not restrict the theory of integration to the subsets of ther eal line, but can be developed on any set to gether with a sigma algebra on it. Thus, the concept of a measure on a measurable space allows us to have a theory of integration in a very general setting which has immense potential for application to diverse areas of mathematics and its applications.

Although the theory of integration is very vast, the attempt in this course is to introduce the students to this modern subject in a simple and natural manner so that they can pursue the subject further with confidence, and also apply the concepts in other branches of mathematics such as those mentioned in the first paragraph.

This course can be a one semester course of about 45 lectures for the first or second semester of a senior undergraduate or a first year graduate programme in mathematics. If it is to be taught for the final year of undergraduate program in mathematics, then the last two chapters, which are on product measure and Fourier transform, can be omitted.

As Lebesgue measure on the real line is introduced in the beginning, no pre-requisite is assumed, except the mathematical maturity to appreciate and grasp concepts in analysis.



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