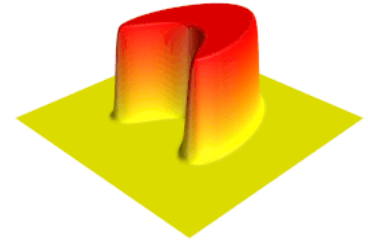


# Partial differential equation (Course Description)

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In mathematics, a **partial differential equation (PDE)** is an equation which computes a function between various partial derivatives of a multivariable function.

The function is often thought of as an "unknown" to be solved for, similar to how  $x$  is thought of as an unknown number to be solved for in an algebraic equation like  $x^2 - 3x + 2 = 0$ . However, it is usually impossible to write down explicit formulae for solutions of partial differential equations. There is correspondingly a vast amount of modern mathematical and scientific research on methods to numerically approximate solutions of certain partial differential equations using computers. Partial differential equations also occupy a large sector of pure mathematical research, in which the usual questions are, broadly speaking, on the identification of general qualitative features of solutions of various partial differential equations, such as existence, uniqueness, regularity and stability. Among the many open questions are the existence and smoothness of solutions to the Navier–Stokes equations, named as one of the Millennium Prize Problems in 2000.



A visualisation of a solution to the two-dimensional heat equation with temperature represented by the vertical direction and color.

Partial differential equations are ubiquitous in mathematically oriented scientific fields, such as physics and engineering. For instance, they are foundational in the modern scientific understanding of sound, heat, diffusion, electrostatics, electrodynamics, thermodynamics, fluid dynamics, elasticity, general relativity, and quantum mechanics (Schrödinger equation, Pauli equation etc.). They also arise from many purely mathematical considerations, such as differential geometry and the calculus of variations; among other notable applications, they are the fundamental tool in the proof of the Poincaré conjecture from geometric topology.

Partly due to this variety of sources, there is a wide spectrum of different types of partial differential equations, and methods have been developed for dealing with many of the individual equations which arise. As such, it is usually acknowledged that there is no "general theory" of partial differential equations, with specialist knowledge being somewhat divided between several essentially distinct subfields.<sup>[1]</sup>

Ordinary differential equations form a subclass of partial differential equations, corresponding to functions of a single variable. Stochastic partial differential equations and nonlocal equations are, as of 2020, particularly widely studied extensions of the "PDE" notion. More classical topics, on which there is still much active research, include elliptic and parabolic partial differential equations, fluid mechanics, Boltzmann equations, and dispersive partial differential equations.

## Table of Contents is in the next pages

# Partial Differential Equations

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# Contents

<b>Contents</b>	<b>i</b>
Preface . . . . .	v
<b>1 Introduction</b>	<b>1</b>
1.1 PDE motivations and context . . . . .	1
1.2 Initial and boundary value problems . . . . .	7
1.3 Classification of equations . . . . .	9
1.4 Origin of some equations . . . . .	13
Problems to Chapter 1 . . . . .	18
<b>2 1-Dimensional Waves</b>	<b>20</b>
2.1 First order PDEs . . . . .	20
Derivation of a PDE describing traffic flow . . . . .	26
Problems to Section 2.1 . . . . .	29
2.2 Multidimensional equations . . . . .	32
Problems to Section 2.2 . . . . .	35
2.3 Homogeneous 1D wave equation . . . . .	36
Problems to Section 2.3 . . . . .	38
2.4 1D-Wave equation reloaded: characteristic coordinates . . .	44
Problems to Section 2.4 . . . . .	49
2.5 Wave equation reloaded (continued) . . . . .	51
2.6 1D Wave equation: IBVP . . . . .	58
Problems to Section 2.6 . . . . .	74
2.7 Energy integral . . . . .	78
Problems to Section 2.7 . . . . .	81
2.8 Hyperbolic first order systems with one spatial variable . . .	85
Problems to Section 2.8 . . . . .	88

<b>3</b>	<b>Heat equation in 1D</b>	<b>90</b>
3.1	Heat equation . . . . .	90
3.2	Heat equation (miscellaneous) . . . . .	97
3.A	Project: Walk problem . . . . .	105
	Problems to Chapter 3 . . . . .	107
<b>4</b>	<b>Separation of Variables and Fourier Series</b>	<b>114</b>
4.1	Separation of variables (the first blood) . . . . .	114
4.2	Eigenvalue problems . . . . .	118
	Problems to Sections 4.1 and 4.2 . . . . .	126
4.3	Orthogonal systems . . . . .	130
4.4	Orthogonal systems and Fourier series . . . . .	137
4.5	Other Fourier series . . . . .	144
	Problems to Sections 4.3–4.5 . . . . .	150
	Appendix 4.A. Negative eigenvalues in Robin problem . . .	154
	Appendix 4.B. Multidimensional Fourier series . . . . .	157
	Appendix 4.C. Harmonic oscillator . . . . .	160
<b>5</b>	<b>Fourier transform</b>	<b>163</b>
5.1	Fourier transform, Fourier integral . . . . .	163
	Appendix 5.1.A. Justification . . . . .	167
	Appendix 5.1.A. Discussion: pointwise convergence of Fourier integrals and series . . . . .	169
5.2	Properties of Fourier transform . . . . .	171
	Appendix 5.2.A. Multidimensional Fourier transform, Fourier integral . . . . .	175
	Appendix 5.2.B. Fourier transform in the complex domain	176
	Appendix 5.2.C. Discrete Fourier transform . . . . .	179
	Problems to Sections 5.1 and 5.2 . . . . .	180
5.3	Applications of Fourier transform to PDEs . . . . .	183
	Problems to Section 5.3 . . . . .	190
<b>6</b>	<b>Separation of variables</b>	<b>195</b>
6.1	Separation of variables for heat equation . . . . .	195
6.2	Separation of variables: miscellaneous equations . . . . .	199
6.3	Laplace operator in different coordinates . . . . .	205
6.4	Laplace operator in the disk . . . . .	212
6.5	Laplace operator in the disk. II . . . . .	216

6.6	Multidimensional equations . . . . .	221
	Appendix 6.A. Linear second order ODEs . . . . .	224
	Problems to Chapter 6 . . . . .	227
<b>7</b>	<b>Laplace equation</b>	<b>231</b>
7.1	General properties of Laplace equation . . . . .	231
7.2	Potential theory and around . . . . .	233
7.3	Green's function . . . . .	240
	Problems to Chapter 7 . . . . .	245
<b>8</b>	<b>Separation of variables</b>	<b>251</b>
8.1	Separation of variables in spherical coordinates . . . . .	251
8.2	Separation of variables in polar and cylindrical coordinates . . . . .	256
	Separation of variable in elliptic and parabolic coordinates . . . . .	258
	Problems to Chapter 8 . . . . .	260
<b>9</b>	<b>Wave equation</b>	<b>263</b>
9.1	Wave equation in dimensions 3 and 2 . . . . .	263
9.2	Wave equation: energy method . . . . .	271
	Problems to Chapter 9 . . . . .	275
<b>10</b>	<b>Variational methods</b>	<b>277</b>
10.1	Functionals, extremums and variations . . . . .	277
10.2	Functionals, Eextremums and variations (continued) . . . . .	282
10.3	Functionals, extremums and variations (multidimensional) . . . . .	288
10.4	Functionals, extremums and variations (multidimensional, continued) . . . . .	292
10.5	Variational methods in physics . . . . .	299
	Appendix 10.A. Nonholonomic mechanics . . . . .	305
	Problems to Chapter 10 . . . . .	306
<b>11</b>	<b>Distributions and weak solutions</b>	<b>312</b>
11.1	Distributions . . . . .	312
11.2	Distributions: more . . . . .	317
11.3	Applications of distributions . . . . .	322
11.4	Weak solutions . . . . .	327
<b>12</b>	<b>Nonlinear equations</b>	<b>329</b>
12.1	Burgers equation . . . . .	329

<b>13 Eigenvalues and eigenfunctions</b>	<b>336</b>
13.1 Variational theory . . . . .	336
13.2 Asymptotic distribution of eigenvalues . . . . .	340
13.3 Properties of eigenfunctions . . . . .	348
13.4 About spectrum . . . . .	358
13.5 Continuous spectrum and scattering . . . . .	365
<b>14 Miscellaneous</b>	<b>370</b>
14.1 Conservation laws . . . . .	370
14.2 Maxwell equations . . . . .	374
14.3 Some quantum mechanical operators . . . . .	375
<b>A Appendices</b>	<b>378</b>
A.1 Field theory . . . . .	378
A.2 Some notations . . . . .	382