Mathematical Modelling

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Chapter 1
Mathematical Modelling

The integration of applications and mathematical modelling into mathematics education plays an important role in many national curricula (Kaiser, 2020; Niss, Blum and Galbraith, 2007), and thus an increasing role in teacher training. Theoretical contributions from mathematics education form the basis of practical-related teacher education. In the following, selected concepts and theoretical backgrounds for modelling are presented and modelling tasks are examined regarding types, categories and criteria. Then, aspects of teaching mathematical modelling are discussed from a theoretical perspective and exemplary results of empirical studies are presented.

1.1 Terms and Definitions Used in Mathematical Modelling

Applications and modelling are indisputably regarded in the international discussion as a relevant part of mathematics education. For example, the *International Conference on the Teaching and Learning of Mathematical Modelling and Applications* (ICTMA) presents the current state of the international debate every two years. Applications and modelling are all aspects of relationships between mathematics and reality, including nature, culture, society and everyday life. In applications, the focus is on the transition from mathematics to reality and primarily on products, while modelling is more about the complementary transition from reality to mathematics and the processes (Niss et al., 2007). Only the term modelling is mentioned hereinafter, but it is always applicable for the transitions in both directions, namely mathematics to reality and from reality to mathematics.

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1.1.1 Mathematical Modelling and Mathematical Model

Over the past few years, the discussion in mathematics education of reality-based teaching has given rise to numerous conceptions about mathematical modelling and the associated translation processes. In preparation for the ICME-3 conference, Werner Blum conducted intensive research of the literature on mathematical modelling. This work later resulted in two-volume documentation of selected literature on application-oriented mathematics education (Kaiser, 2020). This distinguishes between two directions: the so-called *scientifically humanistic* stream, with representatives such as Freudenthal (1973), focuses more on the mathematics processes, while the *pragmatic* stream, with representatives such as Pollak (1968), is characterised more by a utilitarian aim.

In the context of the modelling debate in German, Blum's position (1985) is central, which regards the entire modelling process and the related distinction between mathematics and reality as fundamental to the notion of modelling. For a more in-depth look at the modelling discussion in German, refer to the book by Greefrath and Vorhölter (2016).

The construction of a mathematical model is a characteristic step for modelling. Niss et al. (2007) define the mathematical model term as a mapping: From a realm D of reality, translation processes are made into a subset of the mathematical world M. If the matching mapping rule is called f, a mathematical model can be described by the triple (D, M, f) (Blum & Niss, 1991). Thus, a mathematical model generally consists of defined objects (points, vectors, functions, etc.) that correspond to the elements essential for the initial situation in the real model and certain relationships between these objects that represent the real-world relationships of these elements with each other.

The reason for constructing and using a mathematical model is to understand or process problems from the part of reality. The term "problem" is used here in a broader sense. Thus, the focus is not only on pragmatic application-related problems but also on problems of a more intellectual nature, which are partly aimed at describing, understanding, explaining or even designing parts of the world with their questions (also of a scientific nature) (Niss et al., 2007).

However, the treatment of these problems has natural limits due to the usually inadequate mapping of complex reality with a mathematical model. Since the main focus in the construction of mathematical models is precisely on the possibility of a reduced form of representation in its complexity and a mathematical processing of real data, this incomplete representation is usually quite desirable. Only a certain amount of reality is translated into the mathematical world. Stachowiak (1973) summarised these aspects of the general model concept in three features:

- The mapping feature specifies that a model is a representation of a natural or artificial original.
- The shortening feature is that a model describes only the relevant features of the original, so the model is a reduction.

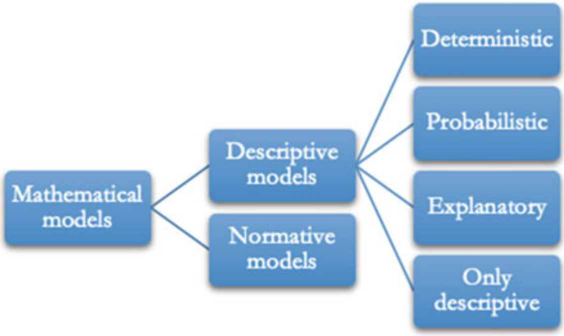

Fig. 1.1 Classification of models by Greefrath and Vorhölter (2016, p. 9)

- The pragmatic feature is that a model always has a specific purpose for certain subjects for a certain period.

Since such simplifications and formalisations are possible in different ways, the corresponding mathematical models also differ (see Fig. 1.1).

For example, prescriptive models are also called *normative* models. An example of this is tax models that set a payroll tax rate at a given gross annual wage. In addition, models can be used as images. These are called *descriptive* models (Freudenthal, 1978). *Explicative* models also provide an explanation for the data or facts and are therefore more meaningful. For example, a model that relates measurements of two variables to one another using linear regression can provide information about the strength of the relationship. *Probabilistic* models, on the other hand, make a prediction. As an example, the Rasch model provides the probability of solving an item with a given personal capability. If the model can determine a future event not only by a probability but by a clear prediction, it is known as a *deterministic* model.

Niss et al. (2007) emphasise that a distinction between mathematical models and the modelling process is particularly important since one or more mathematical models can be constructed during the modelling process. They are therefore integral parts of a larger whole, which is explained in more detail in the following section.

1.1.2 Modelling Processes and Modelling Cycles

The entire process of mathematical modelling can be ideally presented as a cycle, which in turn is formed as a model of the modelling process (Greefrath & Vorhölter, 2016). Until now, a variety of modelling cycles have existed that focus on different aspects (Borromeo Ferri, 2006). The different models are suitable for specific purposes. For example, some are used to illustrate modelling or help learners to work on modelling tasks. Through their extensive theoretical foundations, they represent their own learning content (Greefrath et al., 2013) and serve as a basis for empirical research. An idealised modelling process is described below. The designs are based

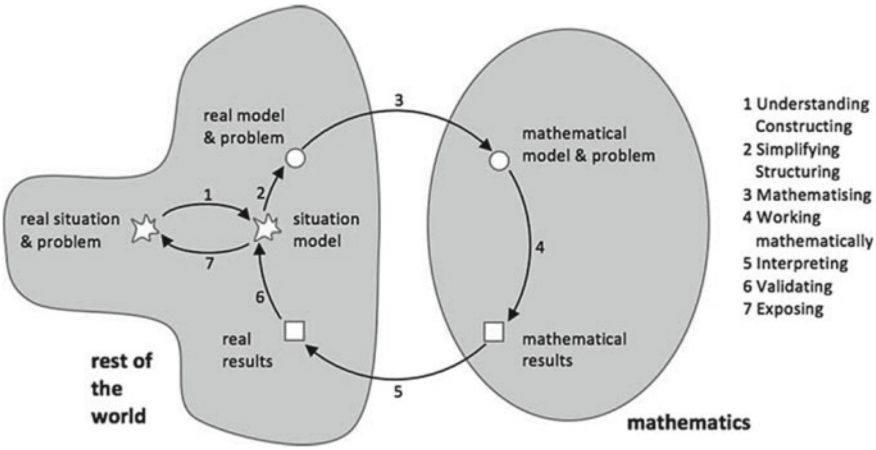

Fig. 1.2 7-Step modelling cycle as per Blum and Leiss (2007, p. 225)

on the 7-step modelling cycle according to Blum and Leiss (2007; see Fig. 1.2), which serves as a basis for the further theoretical considerations in this work.

The starting point for modelling processes is therefore a real-world situation, which involves an authentic problem situation that is processed with mathematical aids. This situation is transferred to a cognitive model according to the knowledge, aims and interests of the modellers. Simplification, structuring and clarification of the resulting mental representation lead to a real model and/or specification of the problem; assumptions must be made and central correlations must be derived. A mathematisation process translates the relevant objects, relationships and assumptions from the real model into mathematics, resulting in a mathematical model that can be used to solve the identified problem (Blum, 2015). Now mathematical methods are used to solve the mathematical problem within the framework of the model created and to obtain a result. The mathematical results thus obtained must then be interpreted in relation to the original real problem context (Greefrath & Vorhölter, 2016). The entire process is then validated. If the solution or the chosen procedure is considered unsatisfactory, individual steps or the entire process must be repeated using a modified or completely different model. Finally, the solution to the original problem of the real world will be outlined and, if necessary, passed on to others (Blum, 2015).

Other idealisations of the modelling process are also conceivable. For example, data acquisition could be considered separately or intermediate steps in the design of the mathematical model could be avoided. Thus, the above cycle is just one of many existing representations of the modelling process. The manifold idealisations of this process can be divided into three groups, which can be characterised by a different number of mathematical steps (Borromeo Ferri, 2006; Greefrath & Vorhölter, 2016). For example, cycles that require only one step from the situation to the mathematical model are assigned to the "direct mathematisation" category. On the other hand, the group of "two-step mathematisation" includes cycles that consider the simplifications in reality, the so-called real model, as an intermediate step from the real situation to the mathematical model.

Since a new perspective that emphasises cognitive analysis, Blum and colleagues developed the modelling cycle shown in Fig. 1.2, which is used to describe the modelling processes of learners as accurately as possible (Blum, 2011). This includes an additional third phase in mathematisation, an individual situation model that is formed from the understanding of the situation by the modellers (Blum & Leiss, 2007).

Real modelling processes of students rarely present the idealised processing steps in linear form. Rather, there are "mini-cycles" or frequent changes between the different stages of the modelling cycle, so-called "individual modelling routes" (Borromeo Ferri, 2018; Galbraith & Stillman, 2006).

All these idealisations have their specific strengths and weaknesses depending on their purpose (Blum, 2015). For cognitive analysis and as a diagnostic tool for (pre-service) teachers, the 7-step modelling cycle depicted seems to be particularly suitable and serves as a basis for further theoretical considerations in the context of the present work (Borromeo Ferri, 2018).

Adequate execution of the modelling processes presented requires certain skills and abilities of the modellers. These modelling competencies are examined in more detail in the following section.

1.1.3 Modelling Competencies

Students should be able to translate between reality and mathematics in both directions and work in the mathematical model. Niss et al. (2007) define modelling competence as follows:

> mathematical modelling competency means the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation, as well as the ability to analyse or compare given models by investigating the assumptions being made, checking properties and scope of a given model etc. (Niss et al., 2007, p. 12)

Promoting the ability to process real-world problems using mathematical aids is thus a central goal of modelling in school.

The above definition describes a so-called global modelling competence in which certain partial processes can be identified by means of an atomistic perspective. Thus, Blum (2015) understands modelling competence as the ability to construct, use or adapt mathematical models by performing the process steps adequately and problem-appropriately, as well as analysing or comparing given models. Modelling competence is therefore not a one-dimensional construct but can be interpreted as a combination of different sub-competencies.

Table 1.1 Sub-competencies of modelling (Greefrath & Vorhölter, 2016, p. 19)

Sub-competency	Description
Understanding	The students construct their own mental model for a given problem situation and thus understand the question
Simplifying	The students separate important and unimportant information about a real situation
Mathematising	The students translate suitably simplified real situations into mathematical models (e.g. term, equation, figure, diagram, function)
Working Mathematically	Students apply heuristic strategies and mathematical knowledge to solve the mathematical problem
Interpreting	The students refer the results obtained in the model to the real situation and thus achieve real results
Validating	The students check the real results in the situation model for adequacy
Exposing	The students refer the answers found in the situation model to the real situation and thus answer the question

The examination of modelling cycles shows a different accentuation of these process steps. In doing so, the ability to execute such sub-processes can be considered as a sub-competency of modelling (Kaiser, 2007; Maaß, 2006; Niss, 2003). These sub-competencies can be characterised in accordance with the 7-step modelling cycle in Fig. 1.2, as shown in Table 1.1.

In addition, metacognitive competences are required for an adequate execution of modelling processes (Stillman, 2011). The absence of metacognition, such as the control of the solution process (Kaiser, 2007) or the reflexion of the adequacy of the solution process (Blomhøj & Højgaard, 2003), can lead to problems during the modelling process.

The question of how modelling processes can be shaped is closely linked to the perspectives on mathematical modelling as well as to the aims pursued with the integration of mathematical modelling into mathematics education. These are considered in more detail in the subsequent section.

1.2 Aims and Perspectives of Mathematical Modelling

The modelling debate showed that different perspectives are taken up—essentially the *scientifically humanistic* as well as the *pragmatic* mainstream (see Sect. 1.1.1). Although these directions have been recognised as the most important currents of debate, the perspectives for mathematical modelling can be more differentiated (Kaiser & Sriraman, 2006), so that great diversity in terms of their understanding of the goal in the field of applications and modelling becomes apparent.

Niss (1996) noted the need to address a discussion of mathematical education and the ways and means of improving its quality, primarily on the basis of a precise and comprehensive formulation of the valid aims and purposes of such education. Only on this basis can the problems of selecting and organising material, teaching methods and qualifications and training of teachers be adequately addressed. Thus, mathematics is

- a powerful tool to understand and master current or future real situations;
- a tool to train general mathematical competences;
- an important part of culture and society as well as the world itself.

Greefrath and Vorhölter (2016) design these general characteristics of mathematics as modelling-specific aims by differentiating between

- *content aims* which take into account the ability of students to recognise and understand real world phenomena;
- *process-oriented aims* that focus on the training of problem solving skills and a general mathematics interest;
- *general aims* that are aimed at building a balanced image of mathematics as a science, responsible participation in society and critical assessment of everyday models, as well as the development of social competences.

Based on comparable considerations, Blum (2015) examines the following justifications for the integration of mathematical modelling into teaching, which he also describes as the aims of teaching and learning applications and modelling:

1. *Pragmatic justification*: Understanding and mastering real-world situations require an explicit engagement with the appropriate application and modelling examples. In these cases, an adequate transfer from intra-mathematical activities cannot be expected.
2. *Formal justification*: General mathematical competences can also be trained through modelling activities. This way, for example mathematical reasoning is further developed by plausibility checks. However, modelling competence can only be acquired by examining the suitable application and modelling examples.
3. *Cultural justification*: Treating real-world phenomena with the aid of mathematics is essential for building a balanced picture of mathematics as a science in a comprehensive sense.
4. *Psychological justification*: Addressing examples from the rest of the world can help to stimulate students' interest in mathematics, demonstrate the relevance of mathematical content, and structure it in a way that promotes understanding.

These justifications or aims of teaching and learning applications and modelling require specific types of appropriate modelling examples. Kaiser and Sriraman (2006) distinguish in research different perspectives on mathematical modelling. The starting point of this identification of different theoretical directions in the current modelling discussion was an analysis of historical and current developments in applications and modelling in mathematics education:

- *Realistic* or *applied* modelling focuses on solving real problems and promoting modelling competence. Theoretically, this direction is based on pragmatic approaches to modelling and thus pursues utilitarian goals, in other words, a better understanding of the rest of the world through the application of mathematics (Kaiser & Sriraman, 2006). It focuses on ostensibly authentic, insignificantly simplified problems, for which holistic approaches are usually chosen, which leads to a comprehensive examination of these problems (Greefrath & Vorhölter, 2016).

- *Educational* modelling is a tradition of the so-called integrated approach and thus emphasizes not only content-related but also process-related aims. It is possible to distinguish more precisely between didactic and conceptual modelling (Kaiser et al., 2015). On the one hand, the *didactic* approach is to promote, on the other hand, to structure the learning processes in modelling. The *conceptual* approach focuses on the understanding and development of concepts. Both are focused on teaching didactic and learning-theoretical meta-knowledge (Kaiser & Sriraman, 2006).
- *Contextual* modelling is largely shaped by the "Model-Eliciting Activities" (MEA) approach developed by Lesh and Doerr (2003) in the USA. This stimulates mathematical activities through challenging real-life situations to stimulate modelling activities (Kaiser et al., 2015). The focused subjective and psychological goals are usually pursued by solving text problems (Kaiser and Sriraman, 2006).
- *Epistemological* or *theoretical* modelling is based on the previously described scientific-humanistic approach and thus focuses on theory-oriented aims. In other words, the application of mathematics, in reality, should contribute to the further development of the same (Kaiser & Sriraman, 2006). Thus, the focus is not so much on translation processes between mathematics and the rest of the world, but on real-life situations as intermediaries are used to address inner-mathematical issues and thus to achieve a science-oriented knowledge gain (Kaiser et al., 2015).
- *Socio-critical* or *socio-cultural* modelling pursues educational goals, such as a critical understanding of the surrounding world (Kaiser and Sriraman, 2006). In this perspective, the role of mathematical models or mathematics in society, in general, is emphasised and critically analysed (Kaiser et al., 2015). Thus, neither the modelling process itself nor the corresponding visualisations are in the foreground (Greefrath & Vorhölter, 2016).
- *Cognitive* modelling can be described as a kind of meta-perspective (Kaiser & Sriraman, 2006). It emphasizes the analysis and understanding of cognitive processes during modelling (Greefrath & Vorhölter, 2016). The development of mathematical thought processes through the use of models as mental or even physical images and the emphasis on modelling as a mental process also plays a role (Kaiser & Sriraman, 2006).

Blum (2015) shows that all the goals of learning theoretical considerations for mathematical modelling can be achieved only through high-quality teaching. Applications and modelling are central to the acquisition of mathematical competences, and a major effort must be made to make mathematical modelling accessible to students. However, it turns out that not only learning, but also teaching mathematical modelling in the classroom is cognitively demanding (Burkhardt, 2004; Freudenthal, 1973; Pollak, 1968). Thus, teachers need different skills, mathematical and non-mathematical knowledge, ideas for tasks and for teaching, and appropriate attitudes and beliefs to teach modelling adequately. In addition, overall teaching becomes more open and evaluation more complex. In view of the explained aims and perspectives of mathematical modelling, various task characteristics can be identified, which are used to stimulate the planned modelling processes in the classroom. There is a wide range of more artificial, less realistic tasks, some of which address only a sub-competency of modelling, to comprehensive, authentic modelling projects with a holistic approach. A detailed discussion of the modelling-specific task types, categories and criteria is contained in the following section.

1.3 Modelling Tasks

What is meant by a modelling task can vary greatly depending on the school or research context. To define other types of tasks and to describe the modelling tasks used in this book, criteria and category systems must be formulated to allow the classification of these tasks. However, the types, categories and criteria presented here are not all assignments that can be clearly defined. This allows classifying the tasks into multiple categories or identifying them as mixed forms. Furthermore, the type of processing in the specific teaching situation, as well as the individual requirements of the students, can influence the type of task. In addition, there are different names and different classification systems in the relevant discussion for the analysis of tasks based on criteria. Therefore, the following section is initially limited to selected, more general categories of mathematical tasks that can be considered relevant for the classification of modelling tasks.

1.3.1 General Categories of Tasks

There are several categories that focus on didactic principles or cognitive processes to examine the properties of tasks of designing the learning processes in mathematics teaching in detail. Among other things, a classification scheme, whose categories primarily cover the potential of tasks for cognitive activation of students, was developed (Neubrand et al., 2013). The dimensions are differentiated between

- *mathematical material areas as a content framework* (contents of geometry, arithmetic, algebra and stochastics; level in the curriculum),
- *types of mathematical work as a cognitive framework* (technical, computational, conceptual task);

- *cognitive elements of the modelling cycle* (non-mathematical modelling, internal mathematical work, basic concepts, mathematical text handling, mathematical thinking, mathematical presentation handling and mathematical reasoning); and
- *Solution room* (direction of processing; multiple solutions)

Moreover, the degree of openness is a feature of mathematics task. Open tasks are those that allow multiple approaches (at different levels) or solutions. In this way, considering tasks with different degrees of openness not only allows students to have their own access to the problems (Greefrath e al., 2017), it also supports students in the development of competences and thus leads to a better understanding of and flexibility in the handling of mathematical content (Borromeo Ferri, 2018). There are several categories of open tasks (see, e.g. Bruder, 2003; Maaß, 2010). In this work, the focus is on the classification of openness according to the initial state, transformation, and target state. For example, a task in which the initial state and transformation are unclear, but the target state is clear, can be called a *reversal problem* (Maaß, 2010).

While the classifications that have been considered so far are of a very general nature, the following focuses on categories that are primarily tailored to reality-related tasks.

1.3.2 Task Categories for Realistic Tasks

When dealing with the properties of modelling tasks, you can formulate a variety of special features that they should fulfil. Such criteria can support both the development and the selection of tasks, and teachers with appropriate classification schemes can gain an overview of modelling tasks. For example, Burkhardt (1989) distinguishes between illustrations of mathematical content and reality-related situations, as well as the latter, whether it can be used to process these standard models or whether new models need to be developed. On the other hand, Galbraith (1995) classifies according to the degree of structuring of the application problem at hand as well as the help provided to solve the problem. A segmentation that is widely used in the German-speaking discussion was developed by Kaiser (1995). In its extensive classification scheme, Maaß (2010) considers, in particular, the nature of the relationship with reality and the didactic intentions of modelling activity as specific criteria for modelling tasks.

Regarding the reality of tasks—in addition to a classification within the framework of the classical task types—a more precise characterisation can be made by the categories of authenticity, relevance to life, closeness to life and relevance to students. The concept of authenticity as well as its contribution to the development of modelling competence is an important area of studies, including the creation of a unified and meaningful meaning for the term “authenticity” itself. This challenge has implications on teaching as well as research (Niss et al., 2007). Greefrath et al.

(2017) therefore focus their attention on the prerequisites for problems to be considered authentic. After this, authenticity in the field of mathematical modelling refers to the non-mathematical context as well as the use of mathematics in the corresponding situation. The non-mathematical context must be real and must not have been specifically designed for the mathematics task. However, Vos (2015) points out that authenticity in this sense does not necessarily mean that a situation exists in the original, but authentic tasks can also represent a good replica of a real situation. The use of mathematics in this situation must also be sensible and realistic, and should not be confined to mathematics education. Authentic modelling tasks are therefore problems that belong exactly to an existing field or problem area and are accepted as such by people working there (Niss, 1992). However, the authenticity of tasks does not yet mean that these tasks are relevant to the present or future lives of students.

Blum (1996) focuses on the context of modelling tasks, like classical task types and considers a task-relevant for mathematics education when certain didactic goals can be achieved. In contrast, Burkhardt (1989) classifies tasks according to the interest that students can have in the context. It also distinguishes between problems arising from students' daily lives, those that may be relevant for students in the future, and those that are only close to the students' lives and whose core focus is on mathematics. The question of whether learners actually consider a context to be interesting, closely linked to, or relevant to their daily lives—as mentioned in the introduction—depends not only on the task itself but also on the specific teaching situation and the individual requirements of the learners. For this reason, PISA (OECD, 2003) distinguishes between tasks relating to the area from which their context comes:

> the situation is the part of the students' world in which the tasks are placed. It is located at a certain distance from the students. For OECD/PISA the closest situation is the student's personal life; next school life, work life and leisure, followed by the local community and society as encountered in daily life. Furthest away are scientific situations. Four situation-types will be defined and used for problems to be solved: personal, educational/occupational, public, and scientific. (OECD, 2003, p. 32)

Regarding the focus of the didactic intention pursued with the modelling activity, the modelling process can always be used to analyse real problems. Blomhøj and Højgaard (2003) distinguish between a holistic and an atomistic approach. According to the first approach, all phases of the modelling cycle will be followed in the modelling process. In an atomistic approach, the modelling task addresses individual phases of the modelling process (e.g. mathematisation). Since students can encounter difficulties in many parts of the solution process when performing modelling tasks, increasing the complexity and the level of demands of the processing, a reduction of the task in the atomistic sense can be useful, and can help to specifically promote or accurately diagnose the partial competencies of modelling. Therefore, the sub-stages of the modelling cycle are often also examined and used to categorize modelling tasks corresponding to the sub-competencies shown in Table 1.1 as was done by (Czocher, 2017).

Based on the previous general categories of tasks as well as the literature of frequently mentioned key properties for tasks with a life relevance, a catalogue of criteria for modelling tasks is compiled below, which serves as a basis for further conceptual considerations of this book.

1.3.3 Selected Criteria for Modelling Tasks

Looking back at the modelling-specific task categories, it can be seen together with Maaß (2010) that the nature of the relationship with reality—more precisely the context of the situation, the authenticity and the relevance for students—seems to be very important for an adequate analysis of reality-related tasks. At the interface of the special and general task criteria, the dimension of the cognitive elements of the modelling cycle—in particular the partial steps of modelling—is highlighted as a characteristic examination feature. Further information can be found in Bruder (2003), Maaß (2010) and Greefrath et al. (2017) clear evidence that the openness of a task, in the sense of multiple approaches and solutions (Schukajlow & Krug, 2013), is an essential feature of modelling tasks. Criteria for the development and analysis of modelling tasks are summarised and specified in Table 1.2.

In particular, it can be noted that through their authenticity and close connection to reality, modelling tasks enable students to access mathematics individually and affectively, and through their openness, with different solution approaches at different levels.

In addition to the selection and development of modelling tasks, appropriate support for modelling processes plays an important role in providing an appropriate learning environment. The task of teachers is primarily to diagnose difficulties and to eventually intervene if needed. The following sections provide more detailed insights into selected theoretical aspects of mathematical modelling regarding these two requirements.

Table 1.2 Catalogue of criteria for modelling tasks (Siller & Greefrath, 2020; Wess & Greefrath, 2019)

Criterion	Specification
Reality relation	The problem definition has a non-mathematical factual reference
Relevance	The problem definition is considered by students to be interesting, closely linked to or relevant to their daily lives
Authenticity	The problem definition is authentic with regard to the non-mathematical aspect The problem definition is authentic with regard to the use of mathematics in concrete situation
Openness	The problem definition allows different solutions The problem definition allows approaches at different levels
Promoting sub-competencies	The problem definition promotes cognitive elements in the form of partial competencies of mathematical modelling

1.4 Difficulties in the Modelling Process

In modelling, it is advisable to make it possible to work independently in cooperative learning environments (Maaß, 2005). Further, working in small groups is an appropriate social form for modelling tasks (Clohessy & Johnson, 2017; Ikeda & Stephens, 2001). However, modelling activities are generally cognitively demanding. In particular, studies have shown that the difficulty of modelling tasks can be explained mainly by the inherent complexity of these tasks, measured by the necessary sub-competencies. Thus, every step in the modelling process of students represents a potential cognitive barrier (Galbraith & Stillman, 2006; Stillman, 2011). Taking these sub-steps into account (see Table 1.1), some typical examples of difficulties and errors encountered by students in the processing of modelling tasks are given below:

- Many students already have problems reading and *understanding* the task. This is not only due to a lack of reading skills (Plath & Leiss, 2018), but students have learnt that they can also work on contextual tasks without carefully reading them and understanding the context (Blum, 2015).
- The *simplifying/structuring* and the associated setting of a real model can be identified as a common source of error (Blum, 2015). Not only do students find it difficult when they meet their own assumptions, but also, in part, from a misunderstanding of the question, errors in the structural contexts implied by them (Schukajlow et al., 2012) are shown.
- In particular, the distinction between real and mathematical models is difficult when *mathematising*. This is not always clear, because the processes of developing a real model and a mathematical model are intertwined. In addition, the change from the real world to the mathematical world is a barrier for the learners, especially since this requires mathematical background knowledge (Galbraith & Stillman, 2006).
- Despite a suitable mathematical result, difficulties can arise in *interpreting*, the translation process from mathematics to the rest of the world. Thus, learners often forget what their calculations actually mean and thus have problems identifying the mathematical results with their real counterparts (Galbraith & Stillman, 2006).
- It seems that it is particularly difficult for students to *validate* (Galbraith & Stillman, 2006). For example, some learners believe that validation is the same for each modelling task, or feel that validation is a depreciation of the result. In addition, interpreting and validating terms often cannot be separated from each other in terms of content. However, students do not usually validate their solutions (Blum, 2015).
- The main difficulties encountered in *communicating* are when students try to reconcile unexpected results with the real situation. These unexpected results are often the result of a previous error, which is easily visible by comparing results with others. They are therefore usually no barrier for attentive learners, although this may be the case for less attentive students (Galbraith & Stillman, 2006).

There are a large number of studies on errors, blockages or difficulties in modelling processes (Galbraith & Stillman, 2006; Galbraith et al., 2007; Stillman et al., 2013; Maaß, 2005; Schaap et al., 2011). An overview of potential difficulties in each modelling phase is given in Table 1.3.

Table 1.3 Difficulties in the modelling process (Klock & Siller, 2020)

Categories (modelling steps)	Subcategories (modelling sub-processes)
1. Forming a real-world model	1.1 Fail to understand the context[a, d]
	1.2 Fail to understand the task[b, d]
	1.3 Fail to search for missing information[c, d]
	1.4 Fail to identify relevant variables[a, b]
	1.5 Fail to make meaningful and simplifying assumptions[a, c]
	1.6 Fail to understand a foreign language[e]
	1.7 Calculating without including the context[d]
2. Forming a mathematical model	2.1 Fail to define variables[a]
	2.2 Fail to realise dependencies between variables[a]
	2.3 Fail to use adequate mathematical methods to mathematise[a]
	2.4 Fail to use technologies[a]
	2.5 Fail to understand the situation/mathematics conceptually[d]
	2.6 Fail to understand mathematical contents[e]
3. Working mathematically	3.1 Fail to use adequate formulae[a]
	3.2 Fail to use adequate solution strategies and algorithms[a, d]
	3.3 Fail to use technologies[a]
	3.4 Fail to understand the situation/mathematics conceptually[c]
	3.5 Fail to solve the model due to an excessive complexity[d]
	3.6 Fail to convert units[d]
4. Interpreting	4.1 Fail to identify the correct meaning of aspects of the mathematical results/model[a, d]
	4.2 Fail to answer the question with the help of the mathematical results[d]
5. Validating	5.1 Fail to reconcile (interim-)results with the real situation[a]
	5.2 Fail to identify the influence of constraints/real world aspects on the mathematical results[a]
	5.3 Fail to find opportunities to improve the model[d]
	5.4 Fail to validate including all relevant aspects[d]

(continued)

Table 1.3 (continued)

Categories (modelling steps)	Subcategories (modelling sub-processes)
	5.5 Fail to improve the model, so it fits to the real-world situation[d]

[a]Galbraith and Stillman (2006)
[b]Schaap et al. (2011)
[c]Stillman et al. (2013)
[d]Maaß (2005)
[e]Added by the authors

In addition to difficulties encountered during the modelling phases, the modelling process also involves metacognitive, affective, social or organizational difficulties, for example: the loss of an overview of your own work, a lack of mathematical self-confidence, a disturbed communication in the working group or an unclear formulated modelling task. Teachers need the knowledge about typical difficulties in modelling processes to be able to react quickly and adequately to them in modelling processes. How adequate intervention in modelling processes looks remains an integral part of current research. The following section deals with this aspect.

1.5 Interventions in the Modelling Process

Within mathematics education, there is an intensive discussion of which teaching behaviour is suitable for the most effective teaching of modelling among students (see, e.g. Tropper et al., 2015). Burkhardt (2006) emphasises that, unlike traditional treatment of the rest of the curriculum, the development of modelling competence entails a change in the role of teachers and related new requirements for teachers. Among other things, discussions must be conducted in a non-direct but supportive manner, students must be given sufficient time and confidence to thoroughly explore individual problems and, if necessary, strategic assistance without detailed proposals. Doerr (2007) also points to a changing role for teachers. In this context, she emphasises that teachers must have a broad and deep understanding of the diversity of approaches that students could pursue in the process of modelling. In addition, the teacher's task is to enable students to interpret, explain, justify and evaluate their models. Teacher intervention is an important element of learning process control in mathematical modelling processes, because modelling tasks often have a high degree of openness. One study suggests that an "operational-strategic" teaching that focuses students' independent work in groups could significantly increase students' modelling competence relative to a "direct" or instructional approach (Schukajlow et al., 2012).

De Jong and Lazonder (2014) classify the help they provide to support research-discovering learning processes according to their specificity. This specificity increases with increasing numbering:

1. *Process constraints*: The complexity of the learning process that is being discovered is reduced by reducing the number of possible options that students must include. An example is the division of tasks into manageable sub-tasks.
2. *Performance Dashboard*: The help gives students an overview of their own work process and its quality. The topics will focus on what has been done and how this contributes to the solution of the task. This aid requires the ability of the learner to continue working with this information.
3. *Prompts*: There are time-appropriate hints that remind students to perform a particular action. They tell us what to do, but not how to do it. This aid requires that students be able to perform the action.
4. *Heuristics*: Compared to Prompts, both the indication that an action is to be performed and how it is to be performed is given. This help is used when students do not know when and how to apply an action in the process.
5. *Scaffolds*: Scaffolds structure the solution process by providing all the components necessary to solve the task. This kind of help is used when the learners cannot manage the solution process independently or it is too complicated for the learners.
6. *Direct Presentation of Information*: The assistance consists of a direct instruction on the content. It makes sense to help learners who have a lack of knowledge or are unable to obtain information themselves.

The range of assistance is consistently oriented towards the competences of the learners. If students are able to control their own learning process, less specificity can be provided. Taxonomy should provide orientation on the selection of suitable aids. Lazonder and Harmsen (2016) showed in a meta-study that none of the categories of help described above promises greater learning success. However, providing help has had a median impact on learning success compared to no help (Lazonder and Harmsen, 2016). This research suggests that not so much the nature of assistance as their individual adaptation to the difficulty of the learner is crucial to greater learning success. Leiss (2007, 2010) also describes this aspect in his work.

Leiss (2007) has developed a general model for teacher interventions (see Fig. 1.3) in which he differentiates between three aspects: the basic knowledge, the area and the characteristics of an intervention (Tropper et al., 2015). A diagnosis of the situation (trigger of the intervention, previous interventions, knowledge required to solve the task, students' competence level, time available) and a diagnosis of difficulties (type

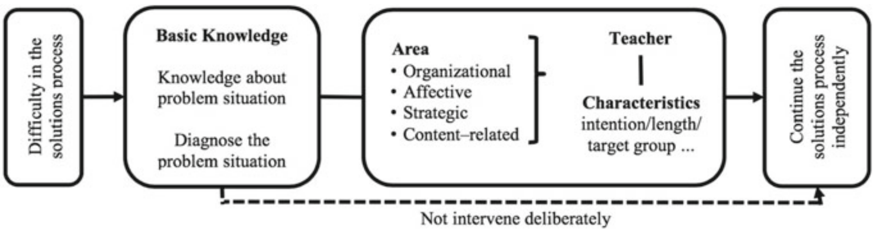

Fig. 1.3 Process model for general teacher interventions (Leiss, 2007; translation by authors)

of difficulty, area and cause of the difficulty, assignment in a theoretical model—here: the modelling cycle) is necessary to create a *basic knowledge* which is crucial for the selection of an adequate intervention.

The *area of intervention* describes to which aspect the intervention refers. Compared to the classification system according to De Jong and Lazonder (2014), these types of intervention have no hierarchy. *Organizational interventions* concern the design of the learning environment ("Watch the time!"). *Affective interventions* influence students' emotional aspects extrinsically ("You can do this!"). *Strategic interventions* are helpful on a meta-level ("What is still missing?" (Blum and Borromeo Ferri, 2010, p. 52)). *Content-related interventions* are related to the concrete contents of the task ("A car consumes 7 L per kilometre."). The area is the central feature of an intervention (Leiss, 2007). In particular, strategic interventions are considered to have a high potential to support students to overcome difficulties in the modelling process (Stender and Kaiser, 2015).

Interventions can be classified by different *characteristics* like the intention of the intervention (statement, question, request), its duration and the addressee (single student, group, whole class) (Leiss, 2007). The aspects of basic knowledge, area and characteristics of an intervention describe a general and idealized intervention. Based on that, Tropper et al. (2015) defined the notion of adaptive teacher interventions which

> are based on a diagnosis of the situation and can be described as an independence-preserving form of support, adapted in form and content to students' learning process, in order to enable them to overcome a (potential) barrier in the process and to continue the process as independently as possible. (Tropper et al., 2015, p. 1226)

Five essential characteristics of adaptive interventions can be identified from this and further definitions (Leiss, 2007; Stender & Kaiser, 2015). In our work adaptive interventions (Klock & Siller, 2019) …

- are based on a diagnosis,
- are adapted in form and content to students' learning process,
- provide minimal help,
- preserve independence,
- have a positive effect on the learning process by overcoming a difficulty.

These aspects are crucial for supporting students in mathematical modelling processes. They are the basis for assessing interventions in terms of adaptivity in our work. Adaptive intervention criteria show that good diagnosis is the basis for adaptive intervention. This is also emphasised in the scaffolding discussion. Van de Pol et al. (2014) distinguish in their process model between a diagnostic part and an intervention part, much like Leiss (2007). As with Leiss, the diagnostic part is preceded by the intervention part (see Fig. 1.4).

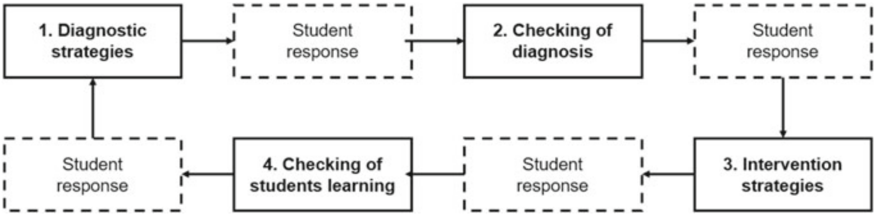

Fig. 1.4 Model of contingent teaching (Van de Pol et al., 2014)

In principle, the examination of the preceding sections shows that teachers play an important role in the development of students' competences and have a decisive influence on the progress of learning (Blum, 2015). To do this, they need different skills, knowledge facets and ideas for tasks and for teaching, as well as appropriate attitudes and beliefs.