Course Description of Measure Theory

What is Measure Theory

In mathematics, a measure on a set is a systematic way to assign a number, intuitively interpreted as its size, to some subsets of that set, called measurable sets. In this sense, a measure is a generalization of the concepts of length, area, and volume. A particularly important example is the Lebesgue measure on a Euclidean space, which assigns the usual length, area, or volume to subsets of a Euclidean spaces, for which this be defined. For instance, the Lebesgue measure of an interval of real numbers is its usual length. Technically, a measure is a function that assigns a non-negative real number or $+\infty$ to (certain) subsets of a set X. A measure must further be countably additive: if a 'large' subset can be decomposed into a finite (or countably infinite) number of 'smaller' disjoint subsets that are measurable, then the 'large' subset is measurable, and its measure is the sum (possibly infinite) of the measures of the "smaller" subsets. In general, if one wants to associate a consistent size to all subsets of a given set, while satisfying the other axioms of a measure, one only finds trivial examples like the counting measure. This problem was resolved by defining measure only on a sub-collection of all subsets; the so-called measurable subsets, which are required to form a σ-algebra. This means that countable unions, countable intersections and complements of measurable subsets are measurable. Non-measurable sets in a Euclidean space, on which the Lebesgue measure cannot be defined consistently, are necessarily complicated in the sense of being badly mixed up with their complement. Indeed, their existence is a non-trivial consequence of the axiom of choice. Measure theory was developed in successive stages during the late 19th and early 20th centuries by Émile Borel, Henri Lebesgue, Johann Radon, and Maurice Fréchet, among others. The main applications of measures are in the foundations of the Lebesgue integral, in Andrey Kolmogorov's axiomatisation of probability theory and in ergodic theory. In integration theory, specifying a measure allows one to define integrals on spaces more general than subsets of Euclidean space; moreover, the integral with respect to the Lebesgue measure on Euclidean spaces is more general and has a richer theory than its predecessor, the Riemann integral. Probability theory considers measures that assign to the whole set the size 1, and considers measurable subsets to be events whose probability is given by the measure. Ergodic theory considers measures that are invariant under, or arise naturally from, a dynamical system.

Course Description

Measure theory is the basis of modern theories of integration, of which Lebesgue integration is a typical case. The theory is quite general, permitting many notions of integration on a broad family of mathematical spaces. It is intimately tied to functional analysis, probability theory, and geometric analysis. This course will provide students with a broad introduction to the subject covering:

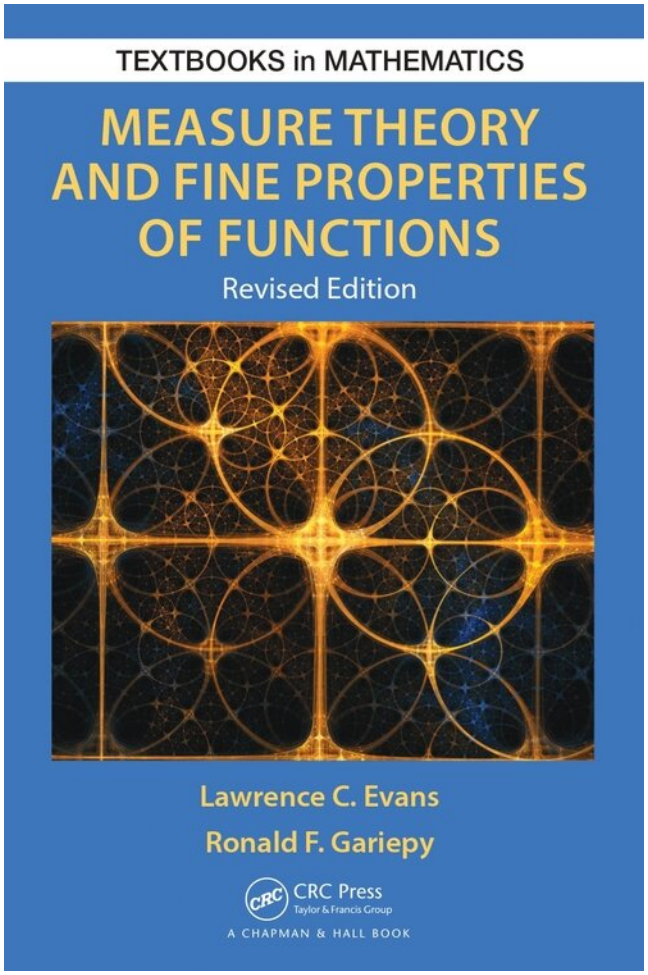
General, Lebesgue and Hausdorff measures; Lusin's and Egoroff's theorems; Limit theorems for integrals; Lp spaces; Product measures and Fubini's theorem; Radon-Nikodym and Riesz theorems; Area and coarea formulas; Sobolev spaces and capacities; BV functions; Sets of finite perimeter; Differentiability and approximation; Probability.

Prerequisites

The only assumed knowledge is the elementary real analysis with proofs. However, some prior exposure to integration, metric space and linear functional analysis would be a definite advantage.

Text Books

Measure theory and fine properties of functions, revised edition by Lawrence C. Evans and Ronald F. Cariepy, Chapman and Hall/CRC, 2015.
Real analysis for graduate students, version 4.3 by Richard F. Bass, 2022.



Real analysis for graduate students

Version 4.3

Richard F. Bass

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Definition

Let X be a set and Σ a σ-algebra over X. A set function μ from Σ to the extended real number line is called a **measure** if the following conditions hold:

- **Non-negativity**: For all $E \in \Sigma$, $\mu(E) \geq 0$.
- $\mu(\varnothing) = 0$.
- **Countable additivity** (or σ-additivity): For all countable collections $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ,

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

If at least one set E has finite measure, then the requirement $\mu(\varnothing) = 0$ is met automatically due to countable additivity:

$$\mu(E) = \mu(E \cup \varnothing) = \mu(E) + \mu(\varnothing),$$

and therefore $\mu(\varnothing) = 0$.

If the condition of non-negativity is dropped, and μ takes on at most one of the values of $\pm\infty$, then μ is called a *signed measure*.

The pair (X, Σ) is called a *measurable space*, and the members of Σ are called **measurable sets**.

A triple (X, Σ, μ) is called a *measure space*. A probability measure is a measure with total measure one – that is, $\mu(X) = 1$. A probability space is a measure space with a probability measure.

For measure spaces that are also topological spaces various compatibility conditions can be placed for the measure and the topology. Most measures met in practice in analysis (and in many cases also in probability theory) are Radon measures. Radon measures have an alternative definition in terms of linear functionals on the locally convex topological vector space of continuous functions with compact support. This approach is taken by Bourbaki (2004) and a number of other sources. For more details, see the article on Radon measures.

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